



## Extension of Goldstein's circulation function for optimal rotors with hub

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*Published in:*  
Journal of Physics: Conference Series (Online)

*Link to article, DOI:*  
[10.1088/1742-6596/753/2/022018](https://doi.org/10.1088/1742-6596/753/2/022018)

*Publication date:*  
2016

*Document Version*  
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*  
Okulov, V., Sørensen, J. N., & Shen, W. Z. (2016). Extension of Goldstein's circulation function for optimal rotors with hub. *Journal of Physics: Conference Series (Online)*, 753, [022018]. <https://doi.org/10.1088/1742-6596/753/2/022018>

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2016 J. Phys.: Conf. Ser. 753 022018

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## Extension of Goldstein's circulation function for optimal rotors with hub

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### Abstract.

The aerodynamic interaction or interference between rotor blades and hub body is usually very complicated, but some useful simplifications can be made by considering the hub with an infinite cylinder. Various attempts to find the optimum distribution of circulation by the lifting vortex lines method have been previously proposed to describe the blade interaction with the hub modeled by the infinite cylinder. In this case, the ideal distribution of bound circulation on the rotor blades is such that the shed vortex system in the hub-area is a set of helicoidal vortex sheets moving uniformly as if rigid, exactly as in the case where there is no influence of the streamtube deformations by the central hub-body. In the present investigation, we consider a more specific problem of the rotor-hub interaction where the initial flow streamtubes and the rotor slipstream submitted strong deformations at the nose-area of the semi-infinite hub.

### 1. Introduction

The classical concept of the optimum rotor with a finite number of blades was formulated in 1919 in the thesis of Betz in accordance with the lifting line theory for a planar wing of Prandtl (see the story in [1-2]). However, at that time, it was not possible to actually solve a problem of helical vortex sheets in the rotor wake, although Goldstein [3] some years later in 1929 showed that a solution indeed could be obtained using an infinite series of Bessel functions. Probably due to the complexity of the series and arriving at a solution to the inverse flow problem, Goldstein's solution is rarely used in propeller design today. Based on the analytical solution to the induction of helical vortex filament developed by Okulov [4], we started to analyze in detail the original formulation of Goldstein [3] and found a more appropriate and simple form of these solutions which has been published in a series of publications by Okulov and Sørensen [5-7]. In the foregoing analysis, it was found that the circulation on the rotor blades falls continuously on the root part of blades, becoming zero only at the axis. This is an acceptable approximation for most aircraft rotors with a very slim hub, but that could not be an apparent solution for the wind turbine in which the hub radius is often quite large and taken 10-20% of the rotor blades. In such cases, the optimum distribution may be different from the Goldstein distribution with zero to the rotor axis.



In the current investigation, an analog of Goldstein's circulation along the blade outside the rotor hub is reconstructed to study the influence of flow disturbances by the nose-area of the hub.

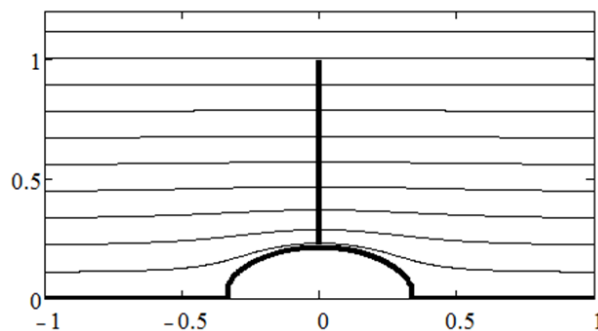
## 2. Model assumptions and hub correction

The aerodynamic interaction or interference between the rotor blades and the hub body is usually very complicated. In this connection, we make some useful simplifications.

A solution to the problem of the ideal distribution of circulation on propeller blades in the presence of the hub is presented in [8] (see also references herein). His assumption is only a continuity condition with a heuristic justification that the Goldstein distribution of circulation on the trailing vortex system remains behind both rotor slipstreams. This simplified treatment employing a continuity relation between the flow behind the classical rotor without the hub ("free-running rotor") and the rotor with the hub concludes an expression between the dimensionless radial coordinate along the blades of both rotors without any knowledge about an influence of flow field deformations generated by the hub. These streamtubes near the hub-nose are adequately represented by a single point source and an equal sink on the rotor axis. Figure 1 shows a cross-section the stream tubes which were designed by the constant values of the velocity potential

$$\psi(r, z) = V_{\infty} z - \frac{Q}{4\pi} \left( \frac{1}{\sqrt{(z-\delta)^2 + r^2}} - \frac{1}{\sqrt{(z+\delta)^2 + r^2}} \right) \quad (1)$$

where  $V_{\infty}$  is the initial velocity of free flow;  $Q$  is a strength of the source and sink;  $(z, r, \theta)$  are cylindrical coordinates.



**Figure 1.** Sketch of the streamtube deformations by the rotor hub replaced by a source and an equal sink.

In our consideration, we will also use the heuristic justification [8] in which the Goldstein's circulation of the free-running rotor replaced by the circulation of the rotor with the hub. These changes of the circulation function base on the stretching of vortex elements in each distinguished streamtubes by their deformations (figure 1) and, as a consequence of Helmholtz' vortex theorem, the bound circulation about a blade element is uniquely related to the circulation at a corresponding radius in infinity in the so-called Trefftz plane without the hub. Assuming the wake to be in equilibrium and neglecting an influence of the hub deformations on the sheet in infinity we can use the original form for the Goldstein circulation

$$G(x, l) = N_b \Gamma(x, l) / hw \quad (2)$$

where the  $N_b$  is a number of blades;  $\Gamma$  – circulation,  $h$  – helical pitch and  $w$  – velocity of the uniform wake translation in the non-disturbance Trefftz plane.

At first, we will reconstruct this initial Goldstein function (2) by representing the trailing vortex sheets in the Trefftz plane by discrete helical vortex filaments with a linear circulation  $\gamma_k$ , employing the analytical model of [4], with strengths adjusted to produce the uniform backward translation [5]. The discrete helical vortex filaments placed in the points  $r_k$ , the induced velocity in point  $r$  is given as follows

$$u_r = \frac{-\gamma_k}{2\pi r} \frac{\sqrt[4]{(l^2 + r^2)(l^2 + r_k^2)}}{l} \operatorname{Im} \left[ \frac{e^{i\chi}}{e^{\mp i\xi} - e^{i\chi}} \pm \frac{l}{24} \left( \frac{2l^2 + 9r_k^2}{(l^2 + r_k^2)^{3/2}} - \frac{2l^2 + 9r^2}{(l^2 + r^2)^{3/2}} \right) \ln(1 - e^{\pm i\xi + i\chi}) \right]$$

$$u_z = \frac{\gamma_k}{2\pi l} \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} + \frac{\gamma_k}{2\pi l} \frac{\sqrt[4]{l^2 + r_k^2}}{\sqrt[4]{l^2 + r^2}} \operatorname{Re} \left[ \frac{\pm e^{i\chi}}{e^{\mp i\xi} - e^{i\chi}} + \frac{l}{24} \left( \frac{3r^2 - 2l^2}{(l^2 + r^2)^{3/2}} + \frac{9r_k^2 + 2l^2}{(l^2 + r_k^2)^{3/2}} \right) \ln(1 - e^{\pm i\xi + i\chi}) \right]$$

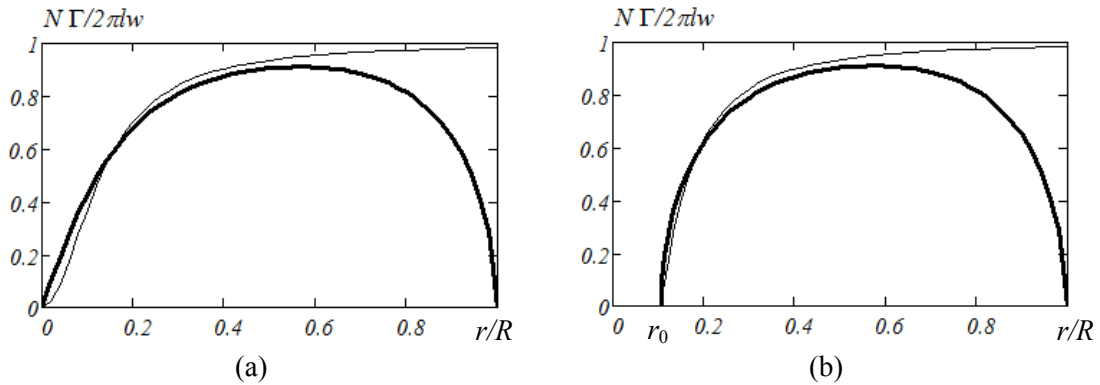
$$u_\theta = (u_0 - u_z)l/r,$$

where  $\chi = \theta - z/l$ ;  $u_0 = \gamma/2\pi l$ ;  $h = 2\pi l$ ;  $e^\xi = \frac{r'}{r_k} = \frac{r}{r_k} \frac{(l + \sqrt{l^2 + r_k^2}) \exp(\sqrt{1 + (r/l)^2})}{(l + \sqrt{l^2 + r^2}) \exp(\sqrt{1 + (r_k/l)^2})}$ .

In the employed notation of “ $\pm$ ” or “ $\mp$ ”, the upper sign corresponds to  $r < r_k$ , and the lower one to  $r > r_k$ . The total velocity induced by the associate discrete vortices of all blades takes form

$$\vec{U}(r, \chi) = \sum_{k=0}^N \sum_{i=0}^{N_b-1} \left\{ \begin{matrix} u_r(\gamma_k, r, r_k, \chi + 2\pi i/N_b) \\ u_z(\gamma_k, r, r_k, \chi + 2\pi i/N_b) \\ u_\theta(\gamma_k, r, r_k, \chi + 2\pi i/N_b) \end{matrix} \right\} \quad (3)$$

in the Trefftz plane, the point  $r_k$  indicates a position of the helical vortex filament with unknown circulation  $\gamma_k$ . Furthermore, a collocation point is placed in the middle of each segment at  $r_j$  and  $\chi = 0$ . There are now  $N+1$  unknown circulations of the discrete helical filaments - and correspondingly, there are  $N$  equations by equating (21) at the  $N$  collocation points. The problem can be closed by the resulting equation with a zero of the total vortex strengths being determined to the Betz rotor condition. To achieve a high accuracy, 100 discrete helical vortex filaments are applied. To validate the model, in [5] the results were compared to the computations by an original simulation of the Goldstein's solution. Figure 2,a shows an example of the initial Goldstein function the non-disturbance Trefftz plane. In addition, the analytical solution of the infinity-bladed rotors under the same conditions was put in the same plots.

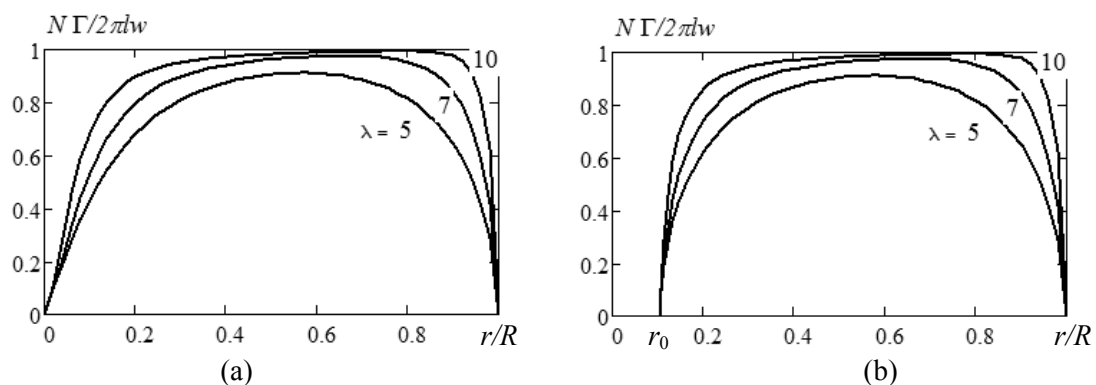


**Figure 2.** (a) Original Goldstein's circulation of 3-bladed (thick lines) or infinity-bladed (thin lines) rotors; (b) a current correction of the circulation for the rotors with the hub (TSR = 5 in both cases)

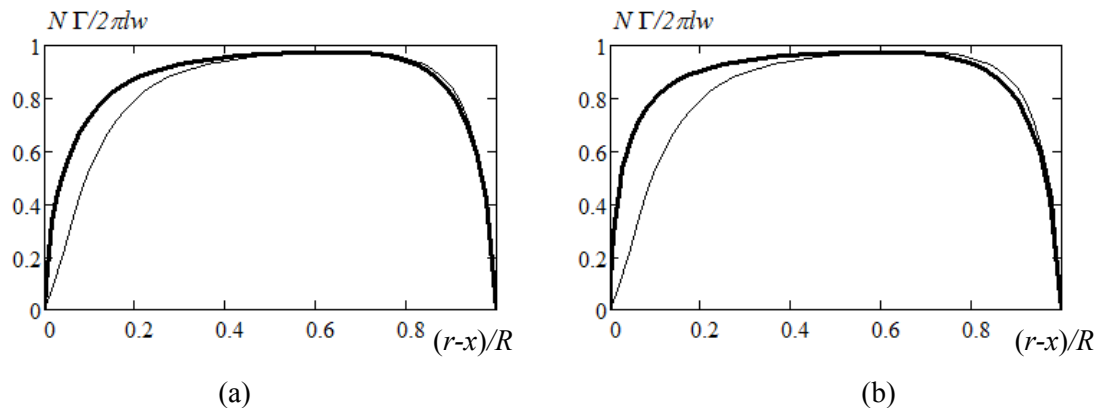
The presence of the hub-body effects on the wake vortex system was simulated separately in each streamtubes and the helicoidal form of the vorticity shedding from the corresponding blade element was fixed to the each tube. We have calculated deformations of the each stream tube by (1) keeping the same circulation to the corresponding fragment of the Goldstein's function (Figure 2,a) coming from the hub-free Trefftz plane. In this approach proposed in [8], we have neglected instability and imperfect helical forms of the total vortex sheets generated by the different initial velocity in the cross-sections of the finite-bladed rotor with the hub.

### 3. Results and conclusions

As a result our first simplified treatment to collect the hub effects by the vortex stretching for each circulation elements under streamtube deformations can be appropriately proved by the slipstream or infinity-bladed rotor theories [1] because the finite-bladed theory [2] should be developed from the requirement that the trailing vortex system has a perfect helicoidal sheet moving as if rigid, exactly as in the case of the free-running rotor without interference from any adjacent body. In accordance with the last remark in the current consideration, we will pay a special attention of a comparison between the heuristic theory for the finite-bladed rotor and the correct consideration for the infinity-bladed rotor. The small difference of the Goldstein functions near the axis (figure 2,a) well-known effect [3, 9] and this effect vanishes when the number of blades tends to infinity. Figure 2,b shows that the hub influence in near hub area removes the difference between both solutions faster.



**Figure 3.** Goldstein's circulation of 3-bladed rotor without hub (a) and with the hub of radius 0.1 for different tip speed ratios  $\lambda$ .



**Figure 4.** Comparison of the original Goldstein's circulation (thin lines:  $x = 0$ ) and our corrections (thick lines:  $x = r_0$ ) along blade length for 3-bladed rotor with hub of radius 0.1 (a) or 0.2 (b) tip speed ratios  $\lambda = 7$

Figures 3a,b show both Goldstein's circulations for different operating regimes of the three-bladed rotor in the real size of the rotor radial direction having, respectively, no hub and hub with 0.1 rotor radius. In Figure 4 we put both distributions of these circulations only along the blade length to clarify the effect of our correction. In the root part of the blade, the original Goldstein's circulation looks smaller than one corrected in accordance with the streamtube deformation by the hub body. This difference grows together with a growth of the hub radius.

We should conclude that for the finite-bladed rotors both original Goldstein's circulation and circulation of the rotor with the hub in the root part (up to the half-length of the blade) of the blades are found in good correlations, respectively, with the analytical solutions of the infinite-bladed rotors. That confirms the heuristic theory of the current study. It was also confirmed that the difference of both solutions grows together with a growth of the hub size and this correction should be taken in the account when the hub size becomes more than a quarter of the rotor size.

### Acknowledgments

The present work was funded in part by the Danish Council for Strategic Research under the project name "OffWind China" (Sagsnr. 0603-00506B) and RSF (grant no. 14-19-00487).

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